Computer Graphics

Section 4.5

Goal: "move" a pixelated object

The motions that we will consider are:

- decreasing or increasing size (scaling)
- rotating the object (rotation)
- moving the object up or down, left or right (translation)

We want to express these motions as matrices.

Example 1: (scaling) Given the vector

$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, increase the x-coordinate by a factor of 4 and decrease the y-coordinate by a factor of 5.

Solution: We want a matrix that sends

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ to } \begin{bmatrix} 3\cdot 4 \\ 5\cdot(1/5) \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix}$$

Matrix sends a 2×1 vector to a 2×1 vector

$$\underset{2\times 2}{A}\ \underset{2\times 1}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 12 \\ 1 \end{bmatrix}}$$

We must produce a 2×2 matrix A

$$\begin{bmatrix} 4 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 4 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 3 + 0 \cdot 5 \\ 0 \cdot 3 + 1/5 \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 1 \end{bmatrix} \checkmark$$

Scaling in General

Given $\begin{bmatrix} x \\ y \end{bmatrix}$, we want to scale x by a factor of c, y by a factor of d. Like our example, the matrix A that effects this scaling must by 2×2. To find its columns, input the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix} \quad 1^{st} \text{ column}$$

$$A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix} \quad 2^{nd} \text{ column}$$

So

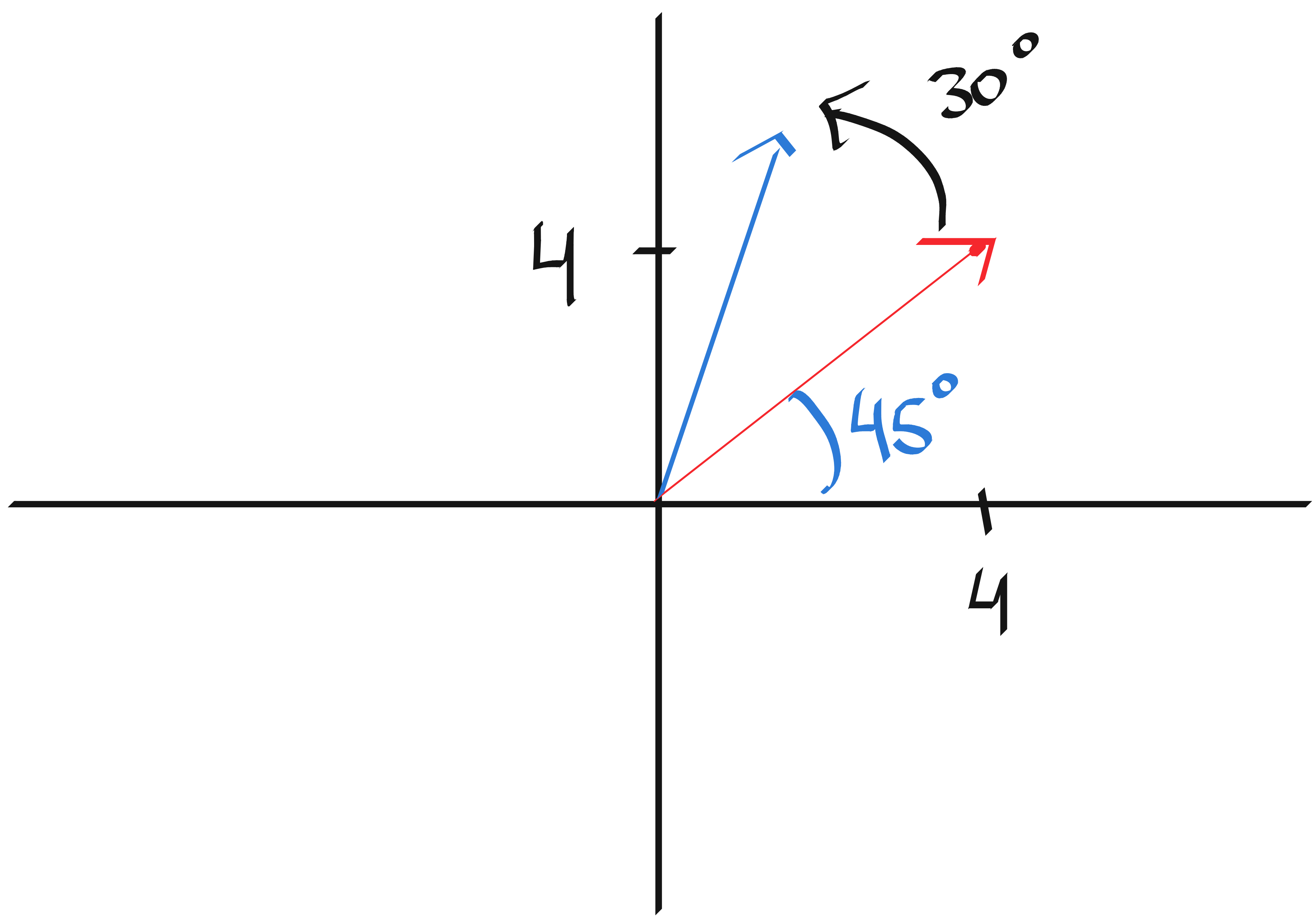
$$A = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$ effects

the scaling.

Example 2: (rotation) Given a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, rotate the vector counterclockwise by $30°$.

Let $x = 4 = y$.

Solution: Rotation looks like



The length of the vector should not change, only the direction it is pointing in should change.

Use $x = r\cos(\theta)$, $y = r\sin(\theta)$ to express our original vector

$$4 = r\cos(\theta) = r\sin(\theta)$$

Since $\theta = 45^\circ$, $\cos(\theta) = \sin(\theta) = \frac{\sqrt{2}}{2}$,

so $4 = r \cdot \frac{\sqrt{2}}{2}$, and

$$r = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

We want

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2}\cos(45^\circ) \\ 4\sqrt{2}\sin(45^\circ) \end{bmatrix} \xrightarrow{30^\circ} \begin{bmatrix} 4\sqrt{2}\cos(45^\circ+30^\circ) \\ 4\sqrt{2}\sin(45^\circ+30^\circ) \end{bmatrix}$$

same magnitude

So the transformation sends

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} \text{ to } \begin{bmatrix} 4\sqrt{2}\cos(75^\circ) \\ 4\sqrt{2}\sin(75^\circ) \end{bmatrix}$$

What are $\cos(75^\circ)$ and $\sin(75^\circ)$, exactly?

Go back to writing $75^\circ = 45^\circ + 30^\circ$

and use the angle addition formulas:

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(75^\circ) = \sin(45^\circ + 30^\circ)$$

$$= \sin(45^\circ)\cos(30^\circ) + \sin(30^\circ)\cos(45^\circ)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\cos(75^\circ) = \cos(45^\circ + 30^\circ)$$

$$= \cos(45^\circ)\cos(30^\circ) - \sin(45^\circ)\sin(30^\circ)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ goes to

$$\begin{bmatrix} 4\sqrt{2}\cos(75^\circ) \\ 4\sqrt{2}\sin(75^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{4}\sqrt{2}\left(\frac{\sqrt{6}+\sqrt{2}}{\cancel{4}}\right) \\ \cancel{4}\sqrt{2}\left(\frac{\sqrt{6}-\sqrt{2}}{\cancel{4}}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 2(\sqrt{3}+1) \\ 2(\sqrt{3}-1) \end{bmatrix} = 2\begin{bmatrix} \sqrt{3}+1 \\ \sqrt{3}-1 \end{bmatrix}$$

Rotation in General

Start with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$

where $r = \sqrt{x^2+y^2}$ and θ is (usually)

$\arctan\left(\frac{y}{x}\right)$ (1^{st} and 4^{th} quadrants).

Want to preserve the magnitude r

and rotate φ degrees counter clockwise.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} \xrightarrow{\varphi^\circ} \begin{bmatrix} r\cos(\theta+\varphi) \\ r\sin(\theta+\varphi) \end{bmatrix}$$

I claim there is a matrix that does this! If there were such a matrix A_φ, then

1^{st} column of $A_\varphi = A_\varphi e_1 = A_\varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

But
$$\begin{array}{c} \text{x coordinate} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \text{y coordinate} \end{array} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$
where $\theta = 0^\circ$ and $r = 1$:

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+0^2} = 1$$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{0}{1}\right)$$
$$= \arctan(0) = 0^\circ.$$

(Matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} a \cdot x + b \cdot y \\ c \cdot x + d \cdot y \end{bmatrix}$$

$$= \begin{bmatrix} a \cdot x \\ c \cdot x \end{bmatrix} + \begin{bmatrix} b \cdot y \\ d \cdot y \end{bmatrix}$$

$$= x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$$

1st column of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$

we set $x=1, y=0$

2nd column of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$x=0, y=1$)

$$A_\varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot \cos(0^\circ + \varphi) \\ 1 \cdot \sin(0^\circ + \varphi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}$$

Similarly, the second column of

x-coordinate

A_φ is $A_\varphi \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, but

y-coordinate

here we need to be careful!

$$r = \sqrt{x^2+y^2} = \sqrt{0^2+1^2} = 1$$

What angle gives the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

We can take $\theta = 90^\circ$.

$$A_\varphi \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A_\varphi \begin{bmatrix} 1 \cdot \cos(90^\circ) \\ 1 \cdot \sin(90^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(90^\circ + \varphi) \\ \sin(90^\circ + \varphi) \end{bmatrix}$$

Using angle addition again,

$$\cos(90^\circ + \varphi) = \cos(\underbrace{\cancel{90^\circ}}_{0}) \cos(\varphi) - \underbrace{\sin(90^\circ)}_{1} \sin(\varphi)$$

$$= -\sin(\varphi)$$

$$\sin(90^\circ + \varphi) = \underbrace{\sin(90^\circ)}_{1} \cos(\varphi) + \sin(\varphi) \underbrace{\cancel{\cos}(90^\circ)}_{0}$$

$$= \cos(\varphi)$$

Then the second column of A_φ is

$$\begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{bmatrix}. \quad \text{So}$$

$$A_\varphi = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Check: $A_\varphi \begin{bmatrix} x \\ y \end{bmatrix} = A_\varphi \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$

$$= \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\varphi)\cos(\theta) - r\sin(\varphi)\sin(\theta) \\ r\cos(\theta)\sin(\varphi) + r\cos(\varphi)\sin(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} r\left(\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)\right) \\ r(\sin(\theta)\cos(\varphi) + \sin(\varphi)\cos(\theta)) \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\varphi) \\ r\sin(\theta+\varphi) \end{bmatrix}$$

again using angle addition.

So

$$A_\varphi = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Example 3: (translation) Can we write the translation that adds 4 to an x-coordinate and subtracts 7 from a y-coordinate as a matrix?

Solution: Suppose there were a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+4 \\ y-7 \end{bmatrix}.$$

If so, then

$$A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

But, as we saw previously,

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$A\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$

So if $x = y = 0$, then

$$A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} a \\ c \end{bmatrix} + 0\begin{bmatrix} b \\ d \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

This shows, via a proof by contradiction, that A cannot be a 2×2 matrix!

Problem: If A is an $m \times n$ matrix,

then

$$\underbrace{A}_{m \times n} \underbrace{\vec{0}_n}_{n \times 1} = \underbrace{\vec{0}_m}_{m \times 1}.$$

That is,

matrices always send zero to zero

Solution: Homogeneous coordinates – used in perspective illustrations.

We identify the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with the vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$. We only consider 3×3 matrices that leave the third coordinate equal to one.

Translations in homogeneous coordinates:

To move a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x+h \\ y+k \end{bmatrix}$, use homogeneous coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

Want to send

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}.$$

The matrix will be

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot x + 0 \cdot y + h \cdot 1 \\ 0 \cdot x + 1 \cdot y + k \cdot 1 \\ 0 \cdot x + 0 \cdot y + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$$

Rotations and Scalings in Homogeneous Coordinates

These work the same way:

if A is a 2×2 matrix,

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, send it to the

3×3 matrix

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} a\cdot x + b\cdot y + 0\cdot 1 \\ c\cdot x + d\cdot y + 0\cdot 1 \\ 0\cdot x + 0\cdot y + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

But

$$\begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix} = \begin{bmatrix} A\begin{bmatrix} x \\ y \end{bmatrix} \\ 1 \end{bmatrix}$$

So the operations of A are preserved.

Example 4: (all three operations)

Write down the 3×3 matrix that, in homogeneous coordinates,

first scales the x-coordinate up by 3 and y-coordinate down by 4, then translates the x-coordinate left by 10 and the y-coordinate up by 8, and finally, rotates the vector by 45° counterclockwise.

$$A = \text{scaling} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \text{translation} = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \text{rotation} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All three, in order indicated:

Multiply $C \cdot B \cdot A$

Reason: you're multiplying on the left of $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

$$C \cdot B \cdot \left(A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right)$$

We get

$$\begin{bmatrix} \frac{3\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} & \sqrt{2} \\ \frac{3\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} & 9\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$